Letters

Comments on "A Procedure Defining Behavior of Weight Functions Near the Edge for Best Convergence Using the Galerkin Method"

Patricio A. A. Laura, Carlos A. Rossit and Diana V. Bambill

The authors are to be congratulated for their important and timely contribution [1]. Their procedure for determining the behavior of the weight functions near the edge so as to accomplish optimum convergence using Galerkin's method for calculating linear functionals of the theory of electromagnetism deserves special credit and constitutes, indeed, an original contribution. The writers feels that their approach will be of considerable interest to researchers in other fields of engineering and applied sciences i.e. heat and mass flow, fracture mechanics, etc.

It appears, also, that their technique will be of value when used in conjunction with universal numerical techniques such as the finite element method.

It is also the purpose of this Letter to briefly refer to a "global" optimization procedure of the coordinate functions used in connection with the Galerkin, Rayleigh–Ritz or the Kantorovich method [2]–[6] based on Lord Rayleigh's optimization procedure [7]. The procedure will be illustrated here when determining eigenvalues in two classical problems:

- determination of the lowest cut-off frequency in a circular waveguide (TM modes)
- calculation of the natural frequencies of vibration of a transducer plate (axisymmetric modes).

The first problem is governed by Helmholtz equation

$$\nabla^2\psi + \lambda^2\psi = 0 \tag{1a}$$

and the boundary condition

$$\psi(r = a) = 0. \tag{1b}$$

Expressing (1a) in terms of the radial variable "r" yields

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \lambda^2\psi = 0 \tag{2}$$

whose exact solution is well-known.

In order to obtain an approximate solution using the Galerkin method one may use the simple approximation

$$\psi \simeq \psi_a = A_1(1 - r^2) \tag{3}$$

and making use of Galerkin's procedure one obtains $\lambda = 2.45$; the exact value being 2.4048.

In view of the fact that Galerkin's method yields upper bounds for the eigenvalues one may use

$$\psi \simeq \psi_a = A_1(1 - r^\gamma), \tag{4}$$

then

$$\lambda = \lambda(\gamma) \tag{5}$$

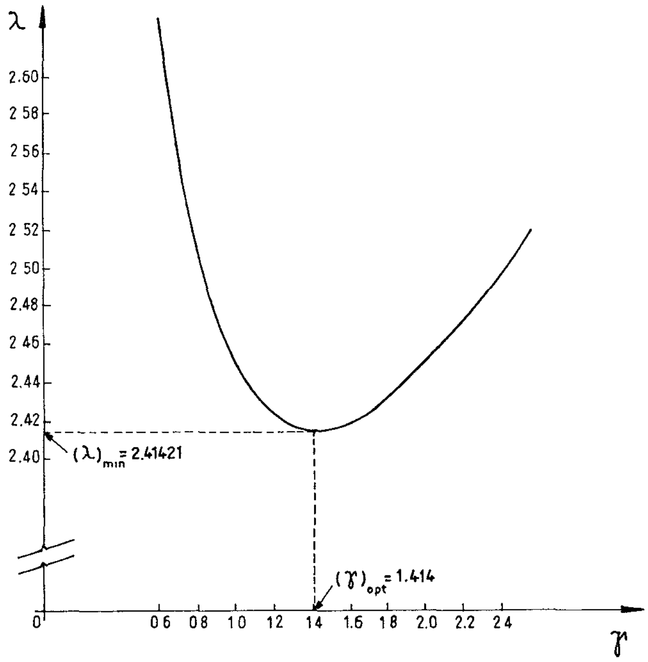

Fig. 1. Determination of the fundamental cut-off frequency coefficient of a circular waveguide (TM modes) using the Galerkin method and a globally optimized coordinate function.

and requiring that λ be a minimum with respect to γ, i.e.

$$\frac{d\lambda}{d\gamma} = 0 \tag{6}$$

one obtains an optimized value of the desired eigenvalue. In this respect expression (4) constitutes a "global" optimization of the coordinate function. For the present case the optimum value of γ is $(\gamma)_{\text{opt}} = 1.414$ and the resulting eigenvalue is $(\lambda)_{\min} = 2.414$ which is in excellent agreement with the exact value (see Fig. 1). Making now

$$\psi \simeq \psi_a = \sum_{i=0}^{3} A_i(1 - r^\gamma)r^i \tag{7}$$

and applying Galerkin's optimized procedure one obtains a fundamental eigenvalue which coincides with the exact one within five significant figures. Minimizing the second and third eigenvalues [5] one achieves also very good accuracy: $\lambda_2 = 5.521$ and $\lambda_3 = 8.68$, the exact values 5.52007 and 8.6537 respectively.

Consider now a free, circular plate of outer radius "a" (this type of problem arises in ultrasonic transducers). The governing differential system is

$$\nabla^2\nabla^2 W - \frac{\rho h}{D}\omega^2 W = 0 \tag{8a}$$

$$\left.\frac{d^2W}{dr^2} + \frac{\mu}{r}\frac{dW}{dr}\right|_{r=a} = 0 \tag{8b}$$

$$\left.\frac{d}{dr}\nabla^2 W\right|_{r=a} = 0 \tag{8c}$$

Manuscript received June 11, 1992; revised December 2, 1992.
The authors are with the Institute of Applied Mechanics (CONICET-SENID-ACCE) and Department of Engineering, Universidad Nacional del Sur, 8000-Bahí ta Blanca, Argentina.
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where ρ: density of the plate material; h: plate thickness; D: flexural rigidity of the plate and μ: Poisson ratio. Poisson's ratio is taken equal to 0.30 for present calculations.

Approximating W by means of

$$W \simeq W_\alpha = \sum_{j=0}^{3} A_j \left[\alpha_j \left(\frac{r}{a}\right)^\gamma + \beta_j \left(\frac{r}{a}\right)^2 + 1\right] \left(\frac{r}{a}\right)^{2j} \tag{9}$$

where the α_j's and β_j's are such that each coordinate function contained in (9) satisfies identically the boundary conditions (8b) and (8c). Applying Galerkin's procedure one obtains, minimizing with respect to γ, that the first, nonzero, eigenvalue is $\sqrt{\rho h/D}\omega_1 a^2 = 9.003$ which coincides with the exact value, within 4 significant figures. The higher eigenvalues are, again, obtained minimizing the higher roots of the frequency equation with respect to γ [5].

It is important to point out that this optimization procedure has also been implemented in finite element codes [8].

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Corrections to "A New Formulation of the Boundary Condition at Infinity for a Hybrid Radiation Mode and Its Application to the Analysis of the Radiation Modes of Microstrip Lines"

Włodzimierz Zieniutycz

In the above paper [1] a few corrections should be introduced as a result of additional analysis and numerical calculation:

- For the odd case ($J_z(\alpha) - \mathrm{odd}$) application of (12) leads to infinite power flux for both solutions of (11). The analysis of behavior of $A_E^{(2)}(\alpha)$ at the points $\alpha = 0$ and $\alpha = \gamma_2$ yields the value of the amplitude which gives the finite power flux of perturbed LSM mode. The proper choice of this amplitude is:

$$A_E^{(2)}(\alpha) \sim \gamma_2^{p+1/2} \tag{1}$$

  where $p > 0$. The perturbed LSE odd solution of (11) shows infinite power flux—this mode has no physical meaning and should be neglected.

- The iterative procedure described in Section III of [1] is divergent for even case ($J_z(\alpha) - \mathrm{even}$). We can, however, rearrange (11) to an alternative set of equations:

$$A_E^{(2)}(\alpha) = f_3[A_E^{(2)}(\alpha), A_H^{(2)}(\alpha)] \tag{2}$$

$$A_E^{(2)}(\alpha) = f_4[A_E^{(2)}(\alpha), A_H^{(2)}(\alpha)] \tag{3}$$

  Spectral amplitude $A_H^{(2)}(\alpha)$ is treated now as a known function and $A_E^{(2)}(\alpha)$ is found by the same iterative procedure (in Fig. 3 [1] we should only replace $A_H^{(2)}(\alpha)$ with $A_E^{(2)}(\alpha)$). The analysis of behaviour of $A_H^{(2)}(\alpha)$ at the points $\alpha = 0$ and $\alpha = \gamma_2$ yields the amplitude which results in the finite power flux of perturbed LSE mode. The proper choice of this amplitude is:

$$A_E^{(2)}(\alpha) \sim \gamma_2^{p+1/2} \tag{4}$$

  where $p > 0$. The second solution (i.e. perturbed LSM even mode) should be neglected as a mode showing infinite power flux. Numerical results of convergence of the proposed procedure are shown in Table I.

In effect we conclude that the hr mode of microstrip line can be treated as a superposition of perturbed LSM odd and LSE even modes. Numerical calculation (see Table I) showed that the modifications did not change the fast convergence of iterative procedure.

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The author is with the Technical University of Gdansk, Faculty of Electronics, Majakowskiego 11/12, 80-952, Gdansk, Poland.
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